THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. III. FEBRUARY, 1896. No. 2.

BIOGRAPHY.

EMILE-MICHEL-HYACINTHE LEMOINE.

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SO EXTENSIVE has become the modern geometry of the triangle that one scarcely realizes that it has almost entirely developed within the last quarter of a century, and that most of its discoverers are still among the living. Lemoine, Brocard, Neuberg, Tucker, and W. J. C. Miller whose mathematical work in the *Educational Times* has done so much for the subject,—these and many others have lived to see their labors crowned with honor by lovers of geometry.

To none of these more than to Emile-Michel-Hyacinthe Lemoine is due the honor of having started this movement, and to him is the following brief sketch devoted.

M. Lemoine was born at Quimper, Finistère, in the west of France, Nov. 22, 1840. His father, a retired captain, who had been in all of the campaigns of the Empire after 1807, placed him as foundation scholar in the military Prytanee of La Flèche, whence he proceeded to the École Polytechnique. He entered this great breeding place of mathematicians at the age of twenty, the year of his father's death, and completed the course in due time. Instead of accepting any of the careers offered by the State to all graduates of the Polytechnic School, M. Lemoine determined to make his own way. Indeed, for the next few years, although engaged in science teaching in Paris, he seems to have run the round of pleasure of which that city is the home *par excellence*. Of great versatility and exceptional conversational powers, with an originality that fascinated and a per-

sonality that impressed his large circle of friends, he lived the life of a *dilettante* in the best sense of the term, and drank at the fountains of pleasure, of politics, of the arts, and of the sciences.

In these days Lemoine led as varied a life in education as in the less scholastic walks. We find him a student in the École des Mines, then *preparateur* of M. Janssen at the École d'Architecture,—supplying the place of his former professor, M. Kiœs, in the preparatory course of the École des Beaux Arts,—perfecting his knowledge of chemistry in the laboratory of Wurtz, for whom he always had a great admiration and between whom and himself there was much affection,—frequenting the courses of the École de Médecine, the hospitals and the clinics,—dabbling in philology,—and ending up by trying the law for a year. This last fancy he was forced to forego because he found himself in disgrace with the Empire through his republican principles and his liberal views on church matters. During these years, too, Lemoine traveled as his income would allow, and when his income failed him he not infrequently traveled as tutor in some wealthy family. Thus it was that he started out in his work as a teacher, full of life and health and hopes, although possibly scattering his attention too much for a career of highest success.

But however the result may have been, an unforeseen accident nipped the experiment in the bud. In 1870, when only a little more than twenty-nine years old, a laryngeal difficulty put an end to his teaching, and required him to leave Paris and seek rest at Grenoble. In the army for a time, he returned to Paris a couple of months after the Commune, and for a number of years filled divers positions in the engineering line. Finally, in 1886, he was appointed city engineer at the head of the gas department, a position which he still holds.

It is, however, with his mathematical work that we are concerned directly. In 1871 he, together with eight or ten other mathematicians, issued the circular which started the Société Mathématique of France. He was among the first to follow and to assist d'Almeida in founding the Journal de Physique and the Société de Physique. He joined with Wurtz, Friedel and others in the organization of the Association Française pour l'Avancement des Sciences. It was while yet a boy in his teens at La Flèche, that, in 1858, he published a short note in the Nouvelles Annales de Mathématiques, which discussed certain properties of the triangle. But it was at the Congrès de Lyon of the Association Française pour l'Avancement des Sciences, in 1873, that he presented his brief but noteworthy paper *Sur quelques propriétés d'un point remarquable de triangle*, and thus, as Casey says, made himself known as the founder of the modern geometry of the triangle. In the same year he published a short note in the Nouvelles Annales on the same subject. In 1874 he presented at the Congrès de Lille a second paper on the geometry of the triangle, entitled *Note sur les propriétés du center des médianes antiparalleles dans un triangle*, a point which has since been quite generally known as the *Lemoine point*, althoueh it is also called the *symmedian point* in England, and the *Grebe point* in Germany. The first paper (1873) contains among others the familiar theorem which may now be

stated thus: "The three parallels to the sides of a triangle through its Lemoine point meet the sides in six concyclic points (the first Lemoine circle)." By the Lemoine (symmedian) point is meant the point of concurrence of the symmedians of a triangle. Since the appearance of these two papers, Lemoine's name has been familiar to all readers of the mathematical journals in every country, and it is for these contributions that he seems destined to be known, rather than for his *Géométrographie* which he considers his greatest work.

La Géométrographie, of which he had the first ideas in 1888, was suggested by him in a memoir, on a more general theme, presented to the Congrès d' Oran of the Association Française pour l'Avancement des Sciences. The title of the paper is *De la mesure de la simplicité dans less Sciences mathématiques*, but for lack of time the study was limited to the simplicity of geometric constructions. On the same subject he published a short note in the *Comptes Rendus* of the Academy for that year,—more strictly *Sur la mesure de la simplicité dans les constructions géométriques*. Since then he has published numerous articles on the same or kindred subjects, in various journals, among them Mathesis (1888), Journal des mathématiques élémentaires (1889), Nouvelles Annales de Mathématiques (1892), in which last named article he considers especially the Problem of Apollonius. Finally, in 1892, at the Congrès de Pau and again at Besançon in 1893 and at Caen in 1894, a series of papers was presented on *La Géométrographie ou l'art des constructions Géométriques*, which may be considered as closing the subject of "geometrography" as applied either to the geometry of the rule and compasses alone, or to those constructions which admit the square, as in descriptive geometry.

Next in importance to the subject of "geometrography," M. Lemoine ranks his work on Continuous Transformation which permits of forming without effort, almost mechanically, a great number of formulæ and theorems relative to tne triangle and to the tetrahedron. The principal memoirs which he has presented on this subject are the following: *Sur les transformations systématiques des formules relatives au triangle*, Congrès de Marseille 1891; *Étude sur une nouvelle transformation dite transformation continue*, in Mathesis for 1891; *Une règle d'analogies dans le triangle et la specification de certaines analogies à une transformation dite transformation continue*, in the Nouvelles Annales for 1893; and finally a memoir entitled *Applications au tétraèdre de la transformation continue.*

Three other geometric studies have been undertaken by M. Lemoine, which deserve especial mention. One is the study of *Triangles Orthologiques.* Steiner demonstrated that if two triangles ABC, $A'B'C'$ are such that the perpendiculars drawn from A, B, C, respectively, on $B'C'$, $C'A'$, $A'B'$ are concurrent, then, reciprocally, the perpendiculars drawn from A', B', C', on BC, CA, AB, respectively, are concurrent. Lemoine calls these triangles *orthologiques* and makes them the basis of a theory developed in several memoirs, notably in one presented at the Congrès de Limoges in 1890. He has also published three papers on the application of geometry to the calculus of probabilities, in the Bulletin de la Sociéte Mathématique (1883), the Nouvelles Annales (1884), and

the proceedings of the Congrès de Grenoble (1885). And finally, there should be mentioned a memoir presented at the Congrès de Nantes in 1875, entitled *Étude systématique du tétraèdre equifacial* (in which the four faces have equal area.)

But in some respects the crowning labor of M. Lemoine is the creation of *L'Intermédiaire des Mathématiciens*, the details of which should be told as a matter of historic interest, especially as they have not heretofore appeared. This publication, although still in its infancy, is known throughout the mathematical world. It consists simply of questions and answers, questions which one asks for information and not for the mere pleasure of displaying some puzzle, questions which bring one into a kind of personal relation to his co-workers whether they be in Russia or South Africa. The idea of the journal is purely M. Lemoine's, and for some time it had been in his mind, but unhappily with no thought of its realization, until the genial influence of a quiet dinner and some good cigars brought about its fruition. M. Laisant had long been a friend of Lemoine's, and it was no uncommon thing for the former to dine with the latter at his home in Rue Littré. On such an occasion, in March, 1893, as they were enjoying a quiet smoke after dinner, the talk ran as usual into mathematics, and Lemoine suggested the idea of the journal. Laisant at once saw the value of the scheme and urged his friend to join him in carrying it out. M. Lemoine replied that it seemed impossible both because he was much occupied with other matters, and because of ill health (from which, unhappily, he is still suffering). Nevertheless, M. Laisant was so persuasive and the influence of the dinner and the cigars so happy that before they separated the project had taken such form that the very next day it was laid before their friend Gauthier-Villars, the great mathematical publisher, and the journal was ushered into being. "Before dinner, nothing could have persuaded me," M. Lemoine writes, "that this idea which I had formed for others would ever be realized by me ; after dinner, the journal was a possibility ; the next day, it was an accomplished fact." Its publication began in January, 1894, and each editor serves during six months of the year.

As one surveys the labors of Lemoine it would seem, from present appearances, that his most valuable work is the foundation of *L'Intermédiaire*, a publication which bids fair to continue for generations because it is really needed. His most original mathematical work seems to be his "geometrography,"—purely a creation of his own, and a contribution which enters into the mathematical work of the military schools of Brussels and Turin, the polytechnic schools of Zurich and Milan, and more or less in many other places. The work which will bring his name to the most readers is his study of the modern geometry of the triangle. In general it may be said that his contribution to geometry has been the very valuable work of showing that the synthetic field is by no means exhausted ; that Euclid left something for this generation to accomplish ; and that an original mind can find abundant material in even so simple a figure as the simplest polygon. How suggestive is this of the vast field which awaits investigators of the more complex geometric figures !

This sketch should not close without a brief reference to the influence that M. Lemoine has exerted in the realm of music. The soirées of M. and Mme. Lemoine are justly celebrated, and each week of the winter sees an assemblage representing the *anciens élevées* of the École Polytechnique, the École Normale, the Marine, and in general a good part of the scientific, literary, and artistic circles of Paris, to listen to a musical programme as original as the mathematical labors of the host. These soirées have exerted a great influence in a musical way, the type which they have fixed being adopted by many societies in and about Paris. One amusing feature of these meetings is the name which designates them. If the writer may be pardoned a personal allusion, he once attended an examination in the École Polytechnique by M. Hermann Laurent. It was one of the most severe he had ever seen,—an exceptionally bright young man submitted to an oral examination that would certainly have floored most American professors,—the examiner, a dyspeptic looking man as cold and as keen as steel and apparently as unsympathetic as ice, though in reality one of the most genial of men. To this justly celebrated mathematician, M. Laurent, is due the name of M. Lemoine's soirées, "La Trompette." Long ago he one day remarked to M. Lemoine in a jesting way, as the latter was excusing himself to attend one of his musical reunions, "Stay here with me, let the trumpet alone." Struck by the name, Lemoine adopted it, and *La Trompette* has ever since designated the delightful soirées with which the Paris cultured world is familiar.

A final word concerning the modesty of M. Lemoine. He estimates his position exactly. He says that he is not a mathematician. He has no claim to rank with Hermite, Poincaré, Picard, Painlevé, Appell, Jordan, Bertrand, Tannery, Darboux, or any of that famous circle which is making Paris such a center of study in the fields of higher modern mathematics. But all mathematicians feel that he has done a noteworthy work in other lines, and for this his name will be known and prominently known in the history of mathematics.

Ypsilanti, Michigan, March, 1896.

WHERE MATHEMATICIANS ARE NEEDED.

By ERIC DOOLITTLE, A. M., Chicago, Illinois.

There is no study of which the conceptions are more grand, nor of which the theorems are more comprehensive and profound than the study of Physical Astronomy. There is no study affording an application of Pure Mathematics in which the perfect harmony of its various parts is more evident; none in which

reason plays a greater part nor approximation a less one. The beauty and simplicity of its first propositions richly reward the early attention of the student, and in the end he is led to the wonderful theorems of *La Place* on the stability of the solar system and the conditions of its formation; theorems which *Baron Fourier* has justly named the highest which the human intelligence can propose.

It is remarkable that more young mathematicians do not enter this absorbing field. The common impression that it requires an unusual mathematical training is largely erroneous. Such a thorough knowledge of Calculus and Mechanics as is shown by the many contributors of the MONTHLY is fully sufficient. Physical Astronomy demands patient and steadfast work; mere brilliance and versatility can accomplish no more of fundamental importance here than in any other true science.

I would urge upon those who are now fitted to enter this or other like work, the great necessity of concentrating their energies upon it. It should be the one object of every devoted student to perfect and advance his own science. It is to this that his whole work must be directed. To such an one years of fragmentary study, first on one subject and then on another, are utterly wasted.

It is the disastrous mistake of many students that they do not realize how soon study for mere amusement or culture should give place to something higher. They fear, often mistakenly, that they are incapable of beginning work of real importance: instead of arranging then a definite series of studies to prepare themselves, they continue to dissipate their strength and accomplish nothing.

Physical Astronomy is calling in many directions for original work. In this country it is comparatively neglected. There are many who are being attracted by the pleasures of Photography and Spectroscopy, but there are few who realize the field which the Fundamental Astronomy opens to them. It contains many problems of the deepest interest. It is filled with questions whose answer requires, not an expensive observatory, but rather mathematical patience or skill.

Readers of the MONTHLY who are determined to accomplish something may well devote themselves to this science. The certainty of their adding to the sum of human knowledge is here greater than in Pure Mathematics, the reward of faithful work unaccompanied by special genius far more certain. The explanation of the variable stars, of the cause and nature of the sun's peculiar rotation, more complete theories of the satellites and of the figures and attractions of the Heavenly Bodies, the determination of the perturbations of the asteroids and other planets and the causes of the anomalies which occur, and the able discussion of a multitude of observations relating to these and other problems are a very few of the many directions in which original work is needed.

As with any true science, Physical Astronomy requires from those who enter upon it long and patient devotion. Its rewards are not bestowed by

chance, nor are they on that account of less value. It is of little popular interest. Its discoveries are seldom sensational. But its dignity and importance cannot be over-estimated. Of American Astronomers, the names of Hill and Newcomb will go down through the ages: their researches will never lose their importance. And whoever adds to this science is contributing to a knowledge which shall endure forever.

Baron Fourier said of La Place:

"Your successors, gentlemen, will witness the accomplishment of the great phenomena whose laws he discovered. They will observe in the motions of the Moon the changes which he predicted and of which he alone was able to assign the cause. The continued observation of Jupiter's satellites will perpetuate the memory of the inventor of the laws which govern them in their courses. The great inequality of Jupiter and Saturn, running through their long periods, and giving to these bodies new situations will recall without ceasing one of his most astonishing discoveries. These are the titles of a true glory which nothing can extinguish. The spectacle of the heavens will be changed, but at those remote epochs the glory of the inventor will continue forever; the traces of his genius bear the seal of immortality."

Chicago University, February 9th, 1896.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

PROPOSITION XXII. *If two straights AB, CD existing in the same plane stand perpendicular to a certain straight BD; but AC joining these perpendiculars makes with them internal acute angles (in hypothesis of acute angle): I say (Fig. 26) the terminated straights AC, BD have a common perpendicular, and indeed within the limits fixed by the designated points A and C.*

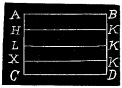


Fig. 26.

Proof. For if AB, CD are equal, it follows (from P. II) that the straight LK, by which these two AC and BD are bisected, will be to them a common perpendicular. But if either be the greater, as suppose AB; let fall to BD (according to Eu. I. 12) from any point L of AC the perpendicular LK, meet-

ing the other BD in K. But it will meet it in some point K existing between the points B and D; otherwise (contrary to Eu. I. 17) the perpendicular LK would cut either AB, or CD, perpendicular to the same BD. So if the angles at the point L are not right, one of them will be acute and the other obtuse. Let the obtuse be toward the point C. But now LK is understood so to proceed toward AB, that it always stands at right angles to BD, and likewise opportunely increased, or diminished, in some point of it cuts the straight AC. It follows that the angles at the intersection points with AC cannot all be obtuse toward the parts of the point C, lest at length in that point A, where the straight LK is congruent with the straight AB, the angle at the point A toward the parts of the point C should be obtuse, when toward these parts it is by hypothesis acute. Since therefore the angle at the point L of this LK is by hypothesis obtuse toward the parts of the point C, the straight LK will not change over in this motion so as to make in some point of it with the straight AC an angle acute toward the parts of the aforesaid point C, unless, before, it changes over so as to make in some point of it with this AC an angle right towards the parts of this same point C. Therefore between the points A, and L will be some one intermediate point H, in which HK perpendicular to this BD is also perpendicular to to the other AC.

In a similar manner is shown to be present a certain XK between LK, CD, which is perpendicular both to the straight BD, and to the straight AC, if namely an angle at the point L is assumed to be obtuse toward the parts of the point A.

It follows therefore that the strights AC, BD will have a common perpendicular, and indeed within the limits fixed by the designated points A, and C, when the joins AB, CD exist in the same plane and are perpendicular to BD.

Quod erat, etc.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from January Number.]

(2) For each generating substitution s_α in a transitive group of degree m_1 find a substitution s_β which (a) interchanges the systems in the same way as s_α interchanges its elements, (b) has its k^{th} power in G_1 where k is the *order* of s_α, *i. e.* the lowest positive value of x which satisfies the equation $s_\alpha^x=1$, and (c) if s_β is

the first substitution which corresponds to a generating substitution in the group of degree m_1, s_β needs only to transform G_1 into itself; otherwise s_β must transform the group already found in the same way as s_α transforms the corresponding part of its group. Continue until all the generating substitutions s_α have been used. We will thus obtain a non-primitive group.

(3) Determine whether the non-primitive group just found is different from each one of those already in the list.

The relation which exists between the required non-primitive group G and the given group of degree m_1 G^1 is called a g_1, 1 *isomorphism*, or a g_1, 1 *correspondence*. The problem of constructing all the non-primitive groups of degree n has its more difficult elements in common with the problem of establishing an a, b correspondence between two groups as may at once be inferred from the given relation. We shall not pursue this subject for the present since only its most evident principles need to be employed when n is small.

To this development of the elementary methods pursued in the construction of non-primitive groups we will add a proof of the general theorem to which we referred in a foot-note. For the sake of simplicity we shall not give the theorem in its most general form.

Theorem. Given that the number of the systems of non-primitivity is n and that the group which does not interchange the systems G_1 is the product of n conjugate transitive groups of which one is found in each system, then there is only one non-primitive group based upon the given G_1 and isomorphic to a transitive group of n elements which is generated by a single substitution.

There certainly is one such group for we may chose s_β so that it will simply permute the systems in the same way as s_α permutes its elements and will have the same order as s_α. Since s_β simply permutes the systems, *i. e.* it permutes the systems as units without permuting the elements of the systems, it must also transform G_1 into itself. Hence G_1 and s_β generate a non-primitive group whenever G_1 differs from identity.

Let t_1^1, t_2^2, ... t_n^n(the upper index standing for the systems in the same order as they are represented by the letters of s_α and the lower index for the particular substitution in the system) be any substitution in the n systems which transform the n constituents of G_1 into themselves. Then will

$$t_1^1\, t_2^2 \,\ldots\ldots t_n^n s_\beta$$

be a symbol for all the substitutions whose degree $\overline{\gtrless}$ the degree of the required group which transform G_1 into itself and permute the systems in the same way as s_α permutes its elements. If this general substitution satisfies the other condition which must be satisfied if it, with the given G_1, generates a non-primitive group we have

$$(t_1^1\, t_2^2 \,\ldots\ldots t_n^n s_\beta\,)^K = \text{some substitution in } G_1,$$

where K is the smallest positive value of x in the equation

$$s_a^x = 1.$$

Since $(t_1^1\, t_2^2 \,\ldots\ldots\, t_n^n s_\beta)^K = t_1^1\, t_2^1 \,\ldots\ldots\, t_n^1\, t_2^2\, t_3^2 \,\ldots\ldots\, t_1^2 \,\ldots\ldots\, t_n^n t_1^n \,\ldots\ldots\, t_{n-1}^n$

we know that $t_1^1\, t_2^2 \,\ldots\ldots\, t_n^n s_\beta$ may be multiplied by some substitution of G_1 so as to give for the new t's

$$t_1^1\, t_2^2 \,\ldots\ldots\, t_n^n = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Consider now the equations

$$(K_1^1\, K_2^2 \,\ldots\ldots\, K_n^n)^{-1} s_\beta\, K_1^1\, K_2^2 \,\ldots\ldots\, K_n^n{}^* =$$

$$\overset{-1}{K}{}_n^n K_1^n\, \overset{-1}{K}{}_{n-1}^{n-1} K_n^{n-1} \ldots\ldots \overset{-1}{K}{}_1^1\, K_2^1\, s_\beta = t_1^1\, t_2^2 \,\ldots\ldots\, t_n^n s_\beta\,.$$

We see directly that the following is a solution of the last equation if (A) is satisfied:

$$K_1^1 = 1,\; K_2^2 = t_1^2\,,\; K_3^3 = t_1^3\, t_2^3\,, \ldots\ldots K_n^n = t_1^n\, t_2^n \,\ldots\ldots\, t_{n-1}^n.$$

Hence all the possible groups are conjugate to the one already given and our theorem is proved. This theorem may be employed with respect to the first subgroups as well as with respect to the entire groups.

In our next paper we shall consider the construction of the third and last class of groups, viz: the *primitive* groups.

[To be Continued.]

ON AN INTERESTING SYSTEM OF QUADRATIC EQUATIONS.

By DR. E. H. MOORE, University of Chicago, and EMMA C. ACKERMANN, Michigan State Normal School.

In C. Smith's Algebra, fourth edition, p. 134, are given for solution, examples 61, 62, 63, which are as follows (the third with a slight modification):

61. The roots of the equation $x^2 + mx + m^2 + a = 0$ are $x_1,\ x_2$; show that $x_1{}^2 + x_1 x_2 + x_2{}^2 + a = 0$.

*This equation follows from the simpler one

$$(ts)^{-1} = s^{-1} t^{-1}$$

and this is true because if we multiply both members by ts we obtain an identity.

62. The roots of the equation $(x^2+1)(a^2+1)-max(ax-1)=0$ are x_1, x_2; show that $(x_1{}^2+1)(x_2{}^2+1)-mx_1x_2(x_1x_2-1)=0$.

63. The roots of the equation $a(x^2+mx+m^2)+bm^2x^2=0$ are x_1, x_2; show that $a(x_1{}^2+x_1x_2+x_2{}^2)+bx_1{}^2x_2{}^2=0$.

The equations possess the following properties: (1), the equation is of the second degree in the variable x and the constant a; (2), the roots x_1, x_2 of the equation are related to each other exactly as are the variable x and constant a.

We seek to generalize these theorems and formulate this problem :

To determine all quadratic equations of the form

$$f(\underset{2}{\smile}\!\!\!\!x, \underset{2}{\smile}\!\!\!\!m)=0,$$

where the function $f(\overset{2}{\smile}{x}, \overset{2}{\smile}{m})$ *is a symmetric function* $f(\overset{2}{\smile}{x}, \overset{2}{\smile}{m})\equiv f(\overset{2}{\smile}{m}, \overset{2}{\smile}{x})$ *of its two arguments* x *and* m *of the second degree in each of them, characterized by the property that between the two roots* x_1, x_2 *which are functions of* m *the relation*

$$f(\overset{2}{\smile}{x_1}, \overset{2}{\smile}{x_2})\equiv 0$$

holds as an identity in m.

I. Let $f(\overset{2}{\smile}{x}, \overset{2}{\smile}{m})\equiv a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0$.

II. $\because f(\overset{2}{\smile}{x_1}, \overset{2}{\smile}{x_2})\equiv 0$, and x_1 and x_2 take the places of x and m, $f(\overset{2}{\smile}{x_1}, \overset{2}{\smile}{x_2})\equiv a+h(x_1+x_2)+bx_1x_2+g(x_1{}^2+x_2{}^2)+f(x_1{}^2x_2+x_2{}^2x_1)+cx_1{}^2x_2{}^2\equiv 0$.

We are to investigate now the conditions on the parameters a, b, c, f, g, h that must hold in order that $f(x_1, x_2)$ may as a function of m be identically 0. The problem then is not necessarily to prove $f(x_1, x_2)\equiv 0$ for all equations, but to find all equations for which it is true that $f(x_1, x_2)\equiv 0$.

III. Let $Kx^2+Lx+M=0$ be the original equation ; x_1 and x_2 the roots ; then $-K(x_1+x_2)=L$; $K(x_1x_2)=M$.

Comparing this equation with I :

$$K\equiv g+fm+cm^2.$$

$$L\equiv h+bm+fm^2.$$

$$M\equiv a+hm+gm^2.$$

IV. Transform equation in I to this form :

$$a+h(x+m)+(b-2g)xm+g(x+m)^2+f(xm)(x+m)+c(xm)^2=0.$$

V. Also equation in II to this form :

$$a+h(x_1+x_2)+(b-2g)x_1x_2+g(x_1+x_2)^2+f(x_1x_2)(x_1+x_2)+c(x_1x_2)^2\equiv 0.$$

VI. Multiply V by K^2:

$$aK^2+hK^2(x_1+x_2)+(b-2g)K^2x_1x_2+gK^2(x_1+x_2)^2$$
$$+fK^2(x_1x_2)(x_1+x_2)+cK^2(x_1x_2)^2=0.$$

VII. VI becomes, by substituting for x_1x_2 and x_1+x_2 their values as given in III:

$$aK^2-hKL+(b-2g)KM+gL^2-fLM+cM^2\equiv 0$$

where K, L, M are given in terms of m in III.

Since VII is an identity in m, the coefficients of the different powers of m are each zero ; ∴ the condition in VII requires that five polynomials homogeneous in a, b, c, f, g, h of degree three shall be zero. Since there are six letters, there are five ratios ; ∴ there are five unknowns in five equations. This system of five cubic equations turns out to be extremely simple.

For, in VII, substituting for K, L, M their values involving m as given in III, collecting terms with reference to m, and using detached coefficients, we have :

$\underbrace{1}$	$\underbrace{m}$	$\underbrace{m^2}$	$\underbrace{m_3}$	$\underbrace{m^4}$	
ag^2	$2afg$	af^2+2acg	$2acf$	ac^2	$\equiv aK^2$
$-gh^2$	$-(fh^2+bgh)$	$-(ch^2+bfh+fgh)$	$-(bch+f^2h)$	$-cfh$	$\equiv -hKL.$
$ag(b-2g)$	$(af+gh)(b-2g)$	$(ac+fh+g^2)(b-2g)$	$(ch+fg)(b-2g)$	$cg(b-2g)$	$\equiv(b-2g)KM$
gh^2	$2bgh$	$b^2g+2fgh$	$2bfg$	f^2g	$\equiv gL^2$
$-afh$	$-(abf+fh^2)$	$-(af^2+bfh+fgh)$	$-(f^2h+bfg)$	$-f^2g$	$\equiv -fLM.$
a^2c	$2ach$	ch^2+2acg	$2cgh$	cg^2	$\equiv cM^2.$

Simplifying and letting c_0, c_1 ...... be coefficient of m^0, m^1 ......

$$c_0\equiv a\{(b-g)g+ac-fh\}=0.$$

$$c_1\equiv 2h\{(b-g)g+ac-fh\}=0.$$

$$c_2\equiv(b+2g)\{(b-g)g+ac-fh\}=0.$$

$$c_3\equiv 2f\{(b-g)g+ac-fh\}=0.$$

$$c_4\equiv c\{(b-g)g+ac-fh\}=0.$$

This means that given $f(x, m)=0$ as in I, then $f(x_1, x_2)\equiv 0$, if, and only if, either $a=2h=b+2g=2f=c=0$, or $(b-g)g+ac-fh=0$; the second alternative is *one* condition, homogeneous, of degree two, between the six homogeneous parameters. Therefore,

All quadratic equations of the form

$$a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0$$

(in which the first member is a symmetric function $f(\overset{2}{x}, \overset{2}{m}) \equiv f(\overset{2}{m}, \overset{2}{x})$ *of its two arguments* x *and* m *of the second degree in each of them), whose parameters are related by the equation*

$$(b-g)g+ac-fh=0,$$

—and, apart from the relatively trivial equation

$$g(x^2-2mx+m^2)=0,$$

only those equations whose parameters are so related—are characterized by the property that between the two roots x_1, x_2 *which are functions of* m *the relation*

$$f(\overset{2}{x_1}, \overset{2}{x_2}) \equiv 0$$

holds as an identity in m.

November 26, 1895.

QUADRATURE OF THE CIRCLE.

By WILLIAM E. HEAL, A. M., Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

The problem of the quadrature of the circle, or what amounts to the same thing, drawing a straight line equal in length to the circumference of a given circle, occupied the attention of mathematicians at a very early date. Long before the time of Archimedes, geometers had attacked the problem with but one result: failure. And for more than twenty centuries mathematicians have been struggling with the problem. Many claimed to have solved it, but their analysis has been, in every case, found to be fatally defective. After centuries of attempt and failure mathematicians began to suspect that the problem might not admit of solution. James Gregory was the first to attempt a proof of the impossibility of the quadrature of the circle. In the opinion of Montucla he succeeded; but later mathematicians have not so decided. Not a score of years have passed since a rigid proof was given that the solution of the problem is really impossible under the conditions usually understood: that is, by the use of the rule and compass only.

It is well known from the geometry that the ratio of the circumference to the diameter of a circle is constant. This constant ratio is usually denoted by the Greek letter π, and it follows at once that if π is a number commensurable with unity that it can be constructed geometrically, and the problem is solved. Lambert, in a memoir presented to the Berlin academy in 1761, was the first to prove that π is incommensurable. Other proofs of this result have been given, especially by Hermite in Crelle's Journal, Vol. 76, which demonstration is reproduced in the Traite de Geometrie of Rouche and Comberousse, 4th edition. But this result, however interesting in itself, does not prove the impossibility of a geometrical construction of π. For example, the square root of 2 (or any non-quadrate number) is incommensurable but is easily constructed geometrically. The first real advance towards the solution of the problem was made by Hermite in 1873. Hermite succeeded in proving that the number e, the base of the Naperian system of logarithms is not only incommensurable but that it can not be a root of a rational algebraic equation of any degree whatever. Such a number is called transcendent. If the number π could be proved to be transcendent the vexed question of the quadrature of the circle would be settled once for all. For this problem requires to derive the number π by a finite number of elementary geometrical constructions. As two straight lines, or a straight line and a circle, or two circles, have not more than two intersections, these processes, or any finite combination of them, can be expressed algebraically in a comparatively simple form ; so that the solution of the problem of the quadrature of the circle would mean that π can be expressed as the root of an algebraic equation solvable by square roots. Hermite did not succeed in proving that π is a transcendent number, but in 1882 Lindemann extended Hermite's proof to include the number π as well as e among the transcendent numbers. Hermite and Lindemann's methods are complicated and obscure and many mathematicians attempted to simplify them. But not until very recently were these attempts rewarded with any degree of success. In January, 1893, Hilbert published a proof of the transcendency of e and π that reduces the problem to such simple terms as to be understood by mathematicians having only a moderate understanding of the principles of the calculus. Hilbert's proof depends upon certain properties of the definite integral

$$\int_0^\infty z^\rho \Big[(z-1)(z-2)(z-3)\ldots\ldots(z-n)\Big]^{\rho+1} e^{-z}dz,$$

suggested by the investigations of Hermite.

Immediately after the publication of Hilbert's proof, Hurwitz published a proof for the transcendency of e based on still more elementary principles. And finally, in May, 1893, Gordan published a proof of the transcendency of e and π in which only the known development of e^x in powers of x is made use of. This last proof is so simple that it should be introduced into university teaching everywhere. The numbers e and π are very intimately related, and before proceeding

to Gordan's proof of the transcendency of these two fundamental numbers I wish to give here a well known proof that e is incommensurable. We have

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots\frac{1}{\underline{|r}}+\frac{1}{\underline{|r+1}}\ldots\ldots$$

Assume, now, that e is a rational number $\frac{a}{r}$, where a and r are integers, and the fraction $\frac{a}{r}$ is in its lowest terms. Multiply this equation by $\underline{|r}$ and we see that all the terms preceding the term $\frac{1}{\underline{|r+1}}$ are integers. The series

$$\frac{1}{r+1}+\frac{1}{(r+1)(r+2)}+\frac{1}{(r+1)(r+2)(r+3)}\ldots\ldots \text{ is less than}$$

$$\frac{1}{r+1}+\frac{1}{(r+1)^2}+\frac{1}{(r+1)^3}\ldots\ldots$$

That is less than $\frac{1}{r}$. Thus we have an integer equal to a proper fraction which is impossible. I will now give Gordan's proof of the transcendency of e and π. The proof for e will be seen to be an extension of the above well-known proof of the irrationality of e and apparently should have been discovered long ago.

The function e^x is defined by the series

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}\ldots\ldots$$

This, if we introduce the symbolic notation

$$\underline{|r}=h^r$$

and multiply by this quantity and any whole number c_r, passes into the form

$$(1) \qquad c_r h^r e^x=c_r(x+h)^r+c_r x^r u_r$$

in which $$u_r=\frac{x}{r+1}+\frac{x^2}{(r+1)(r+2)}\ldots\ldots$$

If $$\mu=\text{mod.}x$$

we have $$\text{mod.}u_r<e^{\mu};$$

and if we put $$u_r=q_r e^{\mu},$$

$$\text{mod.}q_r<1.$$

From (1) it follows:

$$c_r h^r e^x = c_r (x+h)^r + c_r x^r q_r e^{\mu}.$$

And $$e^x \sum_{r=o}^{r=s} c_r h^r = \sum_{r=o}^{r=s} c_r (x+h)^r + e^{\mu} \sum_{r=o}^{r=s} c_r q_r x^r.$$

And if we put $$\sum_{r=o}^{r=s} c_r\, x^r = \phi(x),\ \sum_{r=o}^{r=s} c_r\, q_r\, x^r = \psi(x);$$

(2) $$e^x \phi(h) = \phi(x+h) + e^{\mu} \psi(x).$$

If, now, there is an equation with integral coefficients, satisfied by the number e:

$$\sum_{k=o}^{k=n} c_k e^k = 0,$$

then from (2) we have

(3) $$0 = \sum_{k=o}^{k=n} c_k \phi(k+h) + \sum_{k=o}^{k=n} c_k \psi(k) e^k.$$

If we choose for ϕ the function

$$\phi(x) = \frac{x^{p-1}}{\lfloor p-1} [(x-1)(x-2 \ldots\ldots (x-n)]^p$$

and for ρ a prime number greater than the numbers n and c_0, then will $\phi(k+h)$ in formula (3) become whole numbers.

$$\phi(h+1),\ \phi(h+2) \ldots\ldots \phi(h+n)$$

have the factor p, but

$$c_0 \phi(h)$$

has not. If we let p increase, then ϕ and ψ become as small as we please, and formula (3) is impossible, and the number e transcendent.

If $i\pi$ is a root of an equation with integral coefficients:

(4) $$c(x-w_1)(x-w_2) \ldots\ldots (x-w_{\rho}) = 0,$$

then we have the formula

(5) $$(1+e^{w_1})(1+e^{w_2}) \ldots\ldots (1+e^{w_{\rho}}) = 0.$$

If $c-1$ vanishing quantities are found among the sums

$$w_i\ ;\ w_i+w_k\ ;\ w_i+w_k+w_\lambda \ldots\ldots$$

and we designate those remaining by

$$a_1,\ a_2,\ a_3 \ldots\ldots a_n$$

and their moduli by

$$a_1,\ a_2,\ a_3 \ldots\ldots a_n,$$

formula (5) becomes

$$0=c+\sum_{k=1}^{k=n} e^{a_k}. \tag{6}$$

The symmetric functions of cw_k, as well as those of ca_k, are whole numbers. By formula (2) we have

$$0=c\phi(h)+\sum_{k=1}^{k=n}\phi(a_k+h)+\sum_{k=1}^{k=n} e^{a_k}\psi(a_k). \tag{7}$$

Let $$\phi(x)=\frac{(cx)^{p-1}}{\underline{|p-1}}c^{np}[(x-a_1)(x-a_2)\ldots\ldots(x-a_n)]^p,$$

and let p be a prime number greater than the numbers

$$c\ ;\ n\ ;\ c\ ;\ c^n a_1 a_2 \ldots\ldots a_n.$$

The quantities $\phi(h)$ and $\sum_{k=1}^{k=n}\phi(a_k+h)$ are whole numbers:

$$\sum_{k=1}^{k=n}\phi(a_k+h)$$

contains the factor p, but $c\phi(h)$ does not.

If p increases the moduli of ϕ and ψ become as small as we please.

Formula (7) is impossible, and therefore π is a transcendent number.

THE CENTROID OF AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

It is the object of this paper to put on record, once for all, general values for the centroid of areas, represented by the curve $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the centroid of volumes represented by the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1.$$

I. *Areas.* Let the density vary as $x^{k-1}y^{l-1}$, the thickness being constant.

$$\text{Then } \bar{x}=\frac{\iint x^k y^{l-1}dxdy}{\iint x^{k-1}y^{l-1}dxdy},\quad \bar{y}=\frac{\iint x^{k-1}y^l dxdy}{\iint x^{k-1}y^{l-1}dxdy}.$$

$$\therefore \bar{x}=\frac{\dfrac{a^{k+1}b^l}{\frac{4}{(2m+1)(2n+1)}}\;\dfrac{\Gamma\left\{\frac{k+1}{2}(2m+1)\right\}\Gamma\left\{\frac{l}{2}(2n+1)\right\}}{\Gamma\left\{\frac{k+1}{2}(2m+1)+\frac{l}{2}(2n+1)+1\right\}}}{\dfrac{a^k b^l}{\frac{4}{(2m+1)(2n+1)}}\;\dfrac{\Gamma\left\{\frac{k}{2}(2m+1)\right\}\Gamma\left\{\frac{l}{2}(2n+1)\right\}}{\Gamma\left\{\frac{k}{2}(2m+1)+\frac{l}{2}(2n+1)+1\right\}}}.$$

$$\therefore \bar{x}=\frac{\Gamma(km+m+\frac{k+1}{2})\,\Gamma(km+ln+\frac{k+l}{2}+1)}{\Gamma(km+\frac{k}{2})\Gamma(km+ln+m+\frac{k+l+1}{2}+1)}a\ldots\ldots\ldots\ldots(A).$$

$$\text{Similarly, } \bar{y}=\frac{\Gamma(ln+n+\frac{l+1}{2})\Gamma(km+ln+\frac{k+l}{2}+1)}{\Gamma(ln+\frac{l}{2})\Gamma(km+ln+n+\frac{k+l+1}{2}+1)}b\ldots\ldots\ldots\ldots\ldots\ldots(B).$$

This gives the centroid of a quadrant of the area whatever be the values of k, l, m, n. Let $k=l=1$, so that the density is the same throughout the whole area.

$$\therefore \bar{x}=\frac{\Gamma(2m+1)\Gamma(m+n+2)}{\Gamma(m+\frac{1}{2})\Gamma(2m+n+\frac{5}{2})}a,\ \bar{y}=\frac{\Gamma(2n+1)\Gamma(m+n+2)}{\Gamma(n+\frac{1}{2})\Gamma(m+2n+\frac{5}{2})}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$\therefore \bar{x}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}a=\frac{4a}{3\pi},\ \bar{y}=\frac{\Gamma(1)\Gamma(2)}{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}b=\frac{4b}{3\pi}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore \bar{x}=\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}a=\frac{256a}{315\pi},\ \bar{y}=\frac{\Gamma(3)\Gamma(4)}{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})}b=\frac{256b}{315\pi}.$$

Let $m=n=2$, then $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1$.

$$\therefore \bar{x}=\frac{\Gamma(5)\Gamma(6)}{\Gamma(\frac{5}{2})\Gamma(\frac{17}{2})}a=\frac{2.4.8.12.16.20}{3.5.7.9.11.13.15}\cdot\frac{4a}{\pi},$$

$$\bar{y}=\frac{\Gamma(5)\Gamma(6)}{\Gamma(\frac{5}{2})\Gamma(\frac{17}{2})}b=\frac{2.4.8.12.16.20}{3.5.7.9.11.13.15}\cdot\frac{4b}{\pi}.$$

Let $m=0$, $n=1$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore \bar{x}=\frac{\Gamma(1)\Gamma(3)}{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})}a=\frac{16a}{15\pi},\ \bar{y}=\frac{\Gamma(3)\Gamma(3)}{\Gamma(\frac{3}{2})\Gamma(\frac{9}{2})}b=\frac{128b}{105\pi}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

$$\therefore \bar{x}=\frac{\Gamma(4)\Gamma(5)}{\Gamma(2)\Gamma(7)}a=\frac{a}{5},\ \bar{y}=\frac{\Gamma(4)\Gamma(5)}{\Gamma(2)\Gamma(7)}b=\frac{b}{5},$$ the centroid of the area between the parabola and its tangents as axes.

Let the density vary as xy, so that $k=l=2$.

$$\therefore \bar{x}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(2m+2n+3)}{\Gamma(2m+1)\Gamma(3m+2n+\frac{7}{2})}a,\ \bar{y}=\frac{\Gamma(3n+\frac{3}{2})\Gamma(2m+2n+3)}{\Gamma(2n+1)\Gamma(2m+3n+\frac{7}{2})}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}a=\frac{8a}{15},\quad \bar{y}=\frac{\Gamma(\frac{3}{2})\Gamma(3)}{\Gamma(1)\Gamma(\frac{7}{2})}b=\frac{8b}{15}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}a=\frac{128a}{429},\quad \bar{y}=\frac{\Gamma(\frac{9}{2})\Gamma(7)}{\Gamma(3)\Gamma(\frac{17}{2})}b=\frac{128b}{429}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}a=\frac{2a}{9},\quad \bar{y}=\frac{\Gamma(6)\Gamma(9)}{\Gamma(4)\Gamma(11)}b=\frac{2b}{9}.$$

Let the density vary as x the distance from the axis of ordinates so that $k=2$, $l=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(3m+\frac{3}{2})\Gamma(2m+n+\frac{5}{2})}{\Gamma(2m+1)\Gamma(3m+n+3)}a,\quad \bar{y}=\frac{\Gamma(2n+1)\Gamma(2m+n+\frac{5}{2})}{\Gamma(n+\frac{1}{2})\Gamma(2m+2n+3)}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^{2}+\left(\frac{y}{b}\right)^{2}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(3)}a=\frac{3\pi a}{16},\quad \bar{y}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}b=\frac{3b}{8}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{\Gamma(3)\Gamma(7)}a=\frac{49.45\pi a}{2^{14}},\quad \bar{y}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}b=\frac{63b}{384}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(6)\Gamma(7)}{\Gamma(4)\Gamma(9)}a=\frac{5a}{14},\quad \bar{y}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}b=\frac{3b}{28}.$$

Let the density vary as y the distance from the axis of abscissas so that $k=1$, $l=2$.

$$\therefore\ \bar{x}=\frac{\Gamma(2m+1)\Gamma(m+2n+\frac{5}{2})}{\Gamma(m+\frac{1}{2})\Gamma(2m+2n+3)}a,\quad \bar{y}=\frac{\Gamma(3n+\frac{3}{2})\Gamma(m+2n+\frac{5}{2})}{\Gamma(2n+1)\Gamma(m+3n+3)}b.$$

Let $m=n=0$, then $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(1)\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})\Gamma(3)}a=\frac{3a}{8},\quad \bar{y}=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(3)}b=\frac{3\pi b}{16}.$$

Let $m=n=1$, then $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(3)\Gamma(\frac{11}{2})}{\Gamma(\frac{3}{2})\Gamma(7)}a=\frac{63a}{384},\quad \bar{y}=\frac{\Gamma(\frac{9}{2})\Gamma(\frac{11}{2})}{\Gamma(3)\Gamma(7)}b=\frac{49.45\pi b}{2^{14}}.$$

Let $m=n=\frac{3}{2}$, then $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

$$\therefore\ \bar{x}=\frac{\Gamma(4)\Gamma(7)}{\Gamma(2)\Gamma(9)}a=\frac{3a}{28},\quad \bar{y}=\frac{\Gamma(6)\Gamma(7)}{\Gamma(4)\Gamma(9)}b=\frac{5b}{14}.$$

[To be Continued.]

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

NOTE on the Solution of Problem 53, by J. M. COLAW, A. M., Principal of High School, Monterey, Va.

As originally proposed the problem read "with 7% *annual interest* from date," while, it would seem by inadvertence, as reproduced in the November number, it reads "with *interest at* 7 *per cent.* from date."

I do not find the subject of "Partial Payments on Notes *with Annual Interest*" treated in any of our Arithmetics, except in Olney's *The Science of Arithmetic*, but there are doubtless other exceptions.

On page 191 of *Science of Arithmetic* it is stated that when partial payments are made on notes which bear *Annual Interest*, at other times than those at which the annual interest falls due, the method *usually adopted* is as follows :

Find the interest on the note for 1 year; and find also the amount of the payments made during the year from the times they were severally made to the end of the year.

If the payments amount to more than the interest due, take *their amount* from the amount of the note, and make the remainder a new principal.

But if the amount of the payments does not equal the interest due, the principal remains unchanged, and the amount of the payments is taken from the interest, the remainder being treated as deferred interest.

Proceed in this manner with each year till the time of settlement, the last period being that from the time the last annual interest fell due to the time of settlement.

Mr. Wilke's solution does not follow in all points the rule here laid down as *the usual one.* The question is, what is the rule in Ohio where the note was drawn?

55. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

How long will it take to count a million, in the following manner: the counter is to pronounce each syllable in the names of the successive numbers at the rate of one per second?

Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

One, *two*, ..., *nine*—10 syllables—of the first order, are each pronounced 9 times in every hundred.

$\therefore$ The total for these is $9\times10\times10000=$ 900000.

The same, of the fourth order, are each pronounced 9000 times in every hundred thousand.

∴ The total for these is $9000\times10\times10=$ 900000.

Ten, *eleven*, ..., *nineteen*—20 syllables—of the first and second orders, are each pronounced once in every hundred.

∴ The total for these is $20\times10000=$ 200000.

The same, of the fourth and fifth orders, are each pronounced 1000 times in every hundred thousand.

∴ The total for these is $10\times20\times10000=$ 200000.

Twenty, *thirty*, ..., *ninety*—17 syllables—of the second order, are each pronounced 10 times in every hundred.

∴ The total for these is $10\times17\times10000=$ 1700000.

The same, of the fifth order, are each pronounced 10000 times in every hundred thousand.

∴ The total for these is $10\times17\times10000=$ 1700000.

One hundred, *two hundred*, ..., *nine hundred*—28 syllables—of the third order, are each pronounced 100 times in every thousand.

∴ The total for these is $28\times100\times1000=$ 2800000.

The same, for the sixth order, are each pronounced 100000 times.

∴ The total for these is $28\times100000=$ 2800000.

Thousand is pronounced 999000 times.

∴ The total for this word is 1998000.

The number of syllables in *one million* is 3.

The grand total is 13198003.

∴ 13198003 seconds=152 days, 18 hours, 6 minutes, 43 seconds, the time required.

[Chas. C. Crose, New Windsor, Maryland, sent in a solution of problem 49. The solution is by Algebra and is very good, but as the space in the MONTHLY is very limited even for unsolved problems, we reluctantly omit his solution. The published solution of problem 49 is not valuable because of its brevity, but because each step is the statement of a very elementary mathematical proposition, and hence can be comprehended by any one who has mastered these simple propositions. It is no discredit to a solution to be long if at the same time it is clear in its statements. EDITOR.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by Professor E. W. MORRELL, Department of Mathematics, Montpelier Seminary, Montpelier, Vermont.

Transform $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ into a product.

I. Solution by ROBERT E. MORITZ, B. Sc., Professor of Mathematics in Hastings College, Hastings, Nebraska; and EDGAR KESNER, Boulder, Colorado.

Adding and subtracting $4y^2z^2$, we have

$$(x^4+y^4+z^4-2y^2z^2-2x^2z^2+2y^2z^2)-4y^2z^2=(x^2-y^2-z^2)^2-(2yz)^2.$$

$$(x^2-y^2-z^2)^2-(2yz)^2=(x^2-y^2-z^2-2yz)(x^2-y^2-z^2+2yz)$$
$$=[x^2-(y+z)^2][x^2-(y-z)^2]$$
$$=(x-y-z)(x+y+z)(x-y+z)(x+y-z),$$
$$\text{or } -(y+z-x)(x+y+z)(x-y+z)(x+y-z).$$

Similarly solved by *O. W. ANTHONY, J. SCHEFFER, C. D. SCHMITT, H. C. WILKES, B. F. YANNEY*, and *G. B. M. ZERR.*

II. Solution by A. P. READ, A. M., Clarence, Missouri.

By the method of the last solution, we get

$$[x^2-(y+z)^2][x^2-(y-z)^2]=(x-y-z)(x+y+z)(x-y+z)(x+y-z).$$

In a similar way by adding and subtracting first $4x^2z^2$ and then $4x^2y^2$, we obtain $(y-x-z)(y+x+z)(y-x+z)(y+x-z)$,

and $(z-x-y)(z+x+y)(z-x+y)(z+x-y)$.

Also solved in this way by *M. A. GRUBER.*

III. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Lafayette County, Mississippi.

$x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ may be expressed as the determinant

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}$$

which, as in Burnside and Panton's *Theory of Equations*, second edition, page 253, or, as in Weld's *Theory of Determinants*, pages 41 and 42, may be resolved into the factors

$$-(x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

55. Proposed by MARCUS BAKER, M. A., U. S. Geological Survey, Washington, D. C.

Two right triangles ABC and ABD are so placed as to have one side $x(=AB)$ in common. From P the intersection of their hypotenuses is drawn c perpendicular to x. Knowing the hypotenuses $a=39$ feet and $b=25$ feet, and the perpendicular $c=12\frac{6}{7}$ feet, find x. Note this theorem

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{c} \text{ or } \frac{1}{\sqrt{a^2+x^2}}+\frac{1}{\sqrt{b^2-x^2}}=\frac{1}{c},$$

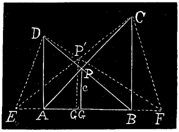
where m and n are the altitudes of the two triangles, respectively. Also find locus of P. Discuss the case when the triangles are general (not right angled).

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science in Texarkana College, Texarkana, Arkansas-Texas.

Let $AB=x$, $PG=c$, $AC=a$, $BD=b$, $CB=m$, $AD=n$.

From the triangles ABC and AGP, we get

$$m:c=x:x-GB\ldots\ldots\ldots\ldots(1).$$



From the triangles ABD and BGP, we get

$$n:c=x:GB\ldots\ldots\ldots\ldots\ldots(2).$$

Eliminating GB between (1) and (2), we get

$$\frac{1}{m}+\frac{1}{n}=\frac{1}{c} \text{ or } \frac{1}{\sqrt{a^2-x^2}}+\frac{1}{\sqrt{b^2-x^2}}=\frac{1}{c}.$$

But $a=39$, $b=25$, $c=12\frac{6}{7}$.

$$\therefore \frac{1}{\sqrt{1521-x^2}}+\frac{1}{\sqrt{625-x^2}}=\frac{7}{90}. \quad \text{Let } 1521-x^2=y^2.$$

$$\therefore \frac{1}{y}+\frac{1}{\sqrt{y^2-896}}=\frac{7}{90}.$$

$$\therefore 7y^4-180y^3-6272y^2+161280y-1036800=0. \quad \therefore y=36,\ x=15.$$

For locus of P, let A be the origin. Using polar co-ordinates, we get

$$\tan\theta=\frac{\sqrt{a^2-x^2}}{x}, \text{ and } \frac{r\sin\theta}{x-r\cos\theta}=\frac{\sqrt{b^2-x^2}}{x}, \text{ for the equations to } AC \text{ and } BD.$$

The value of x from the first in the second gives

$$(a^2-b^2)(r\pm a)^2=(a^4\mp 2a^3r)\sin^2\theta. \quad \text{If } a=b,\ r=\pm\tfrac{1}{2}a.$$

For the general triangle, let $EF=x$, $P'G'=c$, $EC=a$, $DF=b$, $BC=m$, $AD=n$. Then from similar right triangles, we deduce the relation:

$$\frac{c}{n}(x-\sqrt{b^2-n^2})+\frac{c}{m}(x-\sqrt{a^2-m^2})=x.$$

II. Solution by Professor J. SCHEFFER, A. M., Hagerstown, Maryland, and COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Let $BC=m$, $AD=n$, $PG=c$.

Then $$m:c::AB:AG. \quad \therefore AG=\frac{c.AB}{m};$$

$$n:c::AB:BG. \quad \therefore BG=\frac{c.AB}{n}.$$

Adding, $AG+BG=AB=\dfrac{c.AB}{m}+\dfrac{c.AB}{n}$,

$$\text{or } 1=\frac{c}{m}+\frac{c}{n}, \text{ whence } \frac{1}{m}+\frac{1}{n}=\frac{1}{c}.$$

$$m=\sqrt{a^2-x^2}=\sqrt{1521-x^2}\ ;\ n=\sqrt{b^2-x^2}=\sqrt{625-x^2}.$$

Whence $\dfrac{1}{\sqrt{1521-x^2}}+\dfrac{1}{\sqrt{625-x^2}}=\dfrac{7}{90}$. Solving, $x=15$.

[SCHEFFER, SCHMITT.]

Putting $AG=x$, $PG=y$, we find from $\sqrt{b^2-\overline{AB}^2}:y=AB:x$, $\overline{AB}^2=\dfrac{b^2x^2}{x^2+y^2}$; and substituting this in $\dfrac{a^2-\overline{AB}^2}{y^2}=\dfrac{\overline{AB}^2}{(AB-x)^2}$, we get for the Cartesian equation of the locus $\dfrac{(a^2-b^2)x^2+a^2y^2}{y^2(x^2+y^2)}=\dfrac{b^2}{(b-\sqrt{x^2+y^2})^2}$. Changing this into the polar equation by putting $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$, we obtain $r=\dfrac{b\sqrt{a^2-b^2\cos^2\theta}}{b\sin\theta+\sqrt{a^2-b^2\cos^2\theta}}$.

Also solved by *A. H. BELL* and *H. C. WILKES*. [See No. 24, Geometry, Vol. I, page 353, for another solution of a similar problem. Mr. Bell sends a trigonometrical solution, and says that his view of the problem in general is to have given a, b, c, and angles $ABC=BAD=\theta$, to find the base. EDITOR.]

ERRATA.—On page 359, line 16 of December issue, for "t_1+t_1" read t_1+t_2 ; page 360, line 7, in the denominator for "m_5" read m^5 ; page 360, under Case III., for "$-4m^5<n^2$" read $-4m^5>n^2$; in the last line on same page insert $\frac{1}{2}$ before the second radical; and on page 361, line 3, of problem 52, for "(2)" read (3).

PROBLEMS.

62. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

A man raises 1 chicken the first year ; 6, the second ; 35, the third ; 180, the fourth ; 921, the fifth ; 4626, the sixth ; 23215, the seventh ; 116160, the eighth ; and so on. How many does he raise the 20th year, and how many in the twenty years ?

63. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A, B, and C bought unequal shares in 200 acres of land at same price per acre, which they sold for $286.90. A gained as much per cent. on his part as he had acres, B gained $\frac{2}{3}$ as much per cent. on his part as A did, and C lost $9.10 on the cost of his part ; the total net gain was $43\frac{9}{20}$ per cent. How much land did each buy, and what did each receive per acre at the sale ?

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

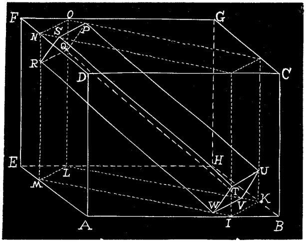
41. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pennsylvania.

Find the length (x) of a rectangular parallelopiped $b=5$ feet, and $h=3$ feet, which can be *diagonally inscribed* in a rectangular parallelopiped $L=83$ feet, $B=64$ feet, and $H=50$ feet.

II. Solution by A. H. BELL, Hillsboro, Illinois, and B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $AB=L=83$ feet, $AE=B=64$ feet, $AD=H=50$ feet, $TUVW=P=$the required inscribed parallelopiped, $VW=b=5$ feet, $WT=h=3$ feet, $BK=z$, $IM=y$, and $WR=TS=UP=VQ=x$.

Then $KH=AM=B-z$, $BI=LE=(b^2-z^2)^{\frac{1}{2}}$, and $AI=HL=L-(b^2-z^2)^{\frac{1}{2}}$.



In the right triangle IAM, $IA^2+AM^2=IM^2$, or

$$[L-(b^2-z^2)^{\frac{1}{2}}]^2+(B-z)^2=y^2 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

In the similar triangles IAM and IBK, we have

$$AI : BK=IM : IK, \text{ or } L-(b^2-z^2)^{\frac{1}{2}} : z=y : b\,;$$

whence $$yz=b[L-(b^2-z^2)^{\frac{1}{2}}] \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Solving (2) for y and substituting its value in (1) we have, after reducing and freeing of radicals,

$$4z^4-4Bz^3+[L^2+B^2-4b^2]z^2+2Bb^2z-(L^2-b^2)b^2=0 \ldots\ldots\ldots(3).$$

Restoring numbers, we have

$$4z^4-2562z^3+10885z^2+3200z-171600=0 \ldots\ldots\ldots\ldots\ldots(4).$$

Solving this equation by *Horner's Method*, we find $z=4$. $\therefore \sqrt{b^2-z^2}=3$.

$\therefore$ From (1) or (2), $y=100$.

If in (3) we let $L=100$, $B=50$, and $b=3$ and solve the equation again for z, we find $z=2.750413+=TI$. $\therefore IW=1.5248$. $\therefore WR=109.4494693746751+$ feet, the required length of the parallelopiped.

Had we solved equation (2) for z and substituted its value in (1), we would have obtained an equation which would give the length of the rectangle $IMLK$, but it would require a great deal of work to free the equation of radicals. We shall now obtain such an equation, or formula.

Let $AB=L$, $BH=B$, $\theta=\angle AIM$, and $x=IM$. Then $AI=x\cos\theta$, $AM=x\sin\theta$, $IB=b\sin\theta$, $BK=b\cos\theta$.

$$\therefore \text{Area of } ABHE=2[\tfrac{1}{2}AI\times AM+\tfrac{1}{2}BI\times BK]+IM\times IK$$

$$=(x^2+b^2)\cos\theta\sin\theta=ab\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also

$$x\cos\theta+b\sin\theta=L\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$x\sin\theta+b\cos\theta=B\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Squaring (2) and (3) and adding the results, we have

$$x^2+c^2+4cx\sin\theta\cos\theta=L^2+B^2\ldots\ldots\ldots\ldots\ldots(4).$$

Equating $\sin\theta\cos\theta$ in (1) and (4), we have, after an easy reduction,

$$x^4-(L^2+B^2+2b^2)x^2+4LBbx-b^2(L^2+B^2-b^2)=0\ldots\ldots\ldots(5),$$

an equation which gives the length of the longest rectangle of given width which can be diagonally inscribed in a given rectangle.

[NOTE.—It is but justice to Mr. Bell to say that he was obliged to protest long and vigorously before he received a proper hearing to his claim that the published solution of Dr. Matz and Mr. Burleson is wrong. It was simply a case of that injustice commonly done to men when we believe them to be wrong and refuse to examine their claims. This problem was proposed a few years ago in the *School Visitor*, and at that time we solved the problem though we did not try to obtain the numerical result. When Dr. Matz and Mr. Burleson sent in their solution, it seemed to us on cursory examination to be obtained on the same plan pursued by us a few years ago. But after Mr. Bell had written to us on several different occasions, we offered to publish his solution that our readers might compare the results. But before doing so, we examined the published solution in the May No. Vol. II and found that it was wrong. The numerical calculation of $z=WR$ is due to Mr. Bell, as is also the last equation and the method of obtaining it. EDITOR.]

49. Proposed by J. C. WILLIAMS, Rome, New York.

Of all triangles inscribed in a given segment of a circle, with the chord as base, the isosceles is the maximum.

I. Solution by M. A. GRUBER, War Department, Washington, D. C., and A. P. REED, Superintendent of Schools, Clarence, Missouri.

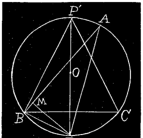
The bases being equal, the maximum triangle is the one having the greatest altitude.

In any segment of a circle, the greatest perpendicular that can be drawn to the chord, is the perpendicular to the middle of the chord. This perpendicular is the altitude of the isosceles triangle.

$\therefore$ The isosceles triangle is the maximum.

II. Solution by J. M. COLAW, A. M., Superintendent of Schools, Monterey, Virginia, and E. KESNER, Boulder, Colorado.

As the segment may be greater or less than a semi-circle, the general proof is for the circle. In the figure it is obvious that the isosceles triangle $P'BC$ is greater than any other triangle ABC, *as its altitude is greater*. Having the given chord as the common base, the *area* depends entirely on the altitude. But the isosceles triangle is a maximum both in *perimeter* and *area*.



Draw PM perpendicular to AB. Then the triangles APM, $P'PB$ are similar, and the diameter $P'P$ is $>AP$; $\therefore$ $P'B$ is $>AM$.

But $2AM=AB+CA$ (*Richardson and Ramsey's Modern Plane Geometry*, *pp.* 24, 131).

$\therefore$ $P'BC$ has the maximum perimeter.

Also solved by *E. L. SHERWOOD* and *G. B. M. ZERR*.

[NOTE.—This problem, with the addition that the isosceles triangle has the maximum perimeter, is Theorem 11, page 131, *Richardson and Ramsey's Modern Plane Geometry*. EDITOR.]

PROBLEMS.

54. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

Prove geometrically :

If through the center of perspective D of a given triangle ABC and its Brocard triangle $A'B'C'$ be drawn straight lines so as to pass through S_a, S_b and S_c (S_a, S_b, and S_c are the middle points of the sides BC, AC, and AB of the triangle ABC) and if S_aD_1 is made equal to DS_a, S_bD_2 equal to DS_b, and S_cD_3 equal to DS_c then are (1) the figures $D_1O'AO$, $D_2O'BO$ and $D_3O'CO$ parallelograms (O and O' are Brocard's points), (2) the triangles $D_1D_2D_3$ and ABC are equal, and (3) D_1A, D_2B, and D_3C intersect in S, (S is the middle point of OO').

55. Proposed by FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Let ab and cd be respectively the major and minor axes of an ellipse, and let α be the angle which a diameter th forms with the major axis ; it is required to find the length of this diameter.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON PROBLEM 26, AVERAGE AND PROBABILITY.

BY G. B. M. ZERR.

In reply to Dr. Martin, for whom I have the utmost respect, I have the following remarks to make. The problem that gives the result $\frac{1}{6}a^2$ is different from the problem that gives the result $\frac{a^2}{2\pi}$. In the former the right angle remains fixed and does not lie on a circle as Dr. Martin states. The problem is as follows: Find the average area of all triangles formed by a straight line of constant length a sliding so that its extremities constantly touch two fixed straight lines at right angles to one another. In the problem under consideration the hypotenuse a is fixed and the right angle moves on the semi-circumference. In the first case the average length of one leg is $\int_0^a xdx \Big/ \int_0^a dx = \frac{1}{2}a$. In the second case the average length is $\int_0^a xds \Big/ \int_0^a ds = a/\pi$. In the first case the average area of all the triangles is $\int_0^a \frac{1}{2}x\sqrt{a^2-x^2}\,dx \Big/ \int_0^a dx = \frac{1}{6}a^2$. In the second case the average area is $\int_0^a \frac{1}{2}x\sqrt{a^2-x^2}\,ds \Big/ \int_0^a ds = \frac{a^2}{2\pi}$, where ds represents an element of arc. It is plainly evident that in the result $\frac{1}{6}a^2$ the leg does not and cannot change its direction or its average length would not be $\frac{1}{2}a$. In the second case it is constantly changing its direction and the right angle is moving on a semicircumference. The problem calls for a given hypotenuse and not one that is constantly changing its direction; hence the result $\frac{a^2}{2\pi}$ is the correct result.

DR. MARTIN'S RESULT IS NOT CORRECT.

F. P. MATZ.

Cause the problem to read: "Find the average area of all right-angled triangles having a given hypotenuse, *if an arm of the triangle vary uniformly;*" then Dr. Martin's result, $\frac{1}{6}h^2$, is perfectly correct.

Strip the problem of this italicised condition; that is, make the problem read as originally proposed; then the number of possible right-angled triangles is proportional to the *length* of the semicircumference of which the given hypotenuse is the diameter. This is *the correct* plan of solution. By *adhering* to *this* plan of solution, the correct result, $h^2/2\pi$, is obtained, regardless as to choice of independent variable.

Dr. Martin's result, $\frac{1}{6}h^2$, is *too great;* for he, by making the number of possible right-angled triangles "proportional to the given hypotenuse," *ignores*

the consideration of the areas of practically an infinitude of right-angled triangles of which the major portion have one *rather small* acute angle—thus giving them areas *smaller* than $\frac{1}{8}h^2$.

Since not only all of Dr. Martin's *ignored* right-angled triangles, but *all possible* right-angled triangles, have been properly averaged in my solutions leading to (the result) $h^2/2\pi$, I repeat that this result is the correct one.

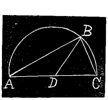
Mechanicsburg, Pa.

A REPLY TO DR. MARTIN'S NOTE.

BY THE EDITOR OF THIS DEPARTMENT.

I will say at first, that I too, have profound regard for Dr. Martin, and his opinion on a subject in which he was the pioneer writer in America should not be assailed simply for the sake of controversy.

His argument is entirely sound as to fact but not as to interpretation. It is true the triangles are *not* uniformly distributed on the semicircumference IF the number of triangles is to be obtained by varying one of the legs of the triangle. That this is true may be easily shown from the figure. Let $AC=a$, the hypotenuse, and $BC=x$, angle $BDC=\theta$. Then $x=a\sin\angle BDC=a\sin\frac{1}{2}\theta$. Differentiating, we have



$\frac{dx}{d\theta}=\frac{1}{2}a\cos\frac{1}{2}\theta=\frac{1}{2}a\sqrt{\frac{1}{2}(\cos\theta+1)}$. $\therefore dx$ increases $\sqrt{\frac{1}{2}(\cos\theta+1)}$ times as fast as $d\theta$. When $\theta=0$, dx and $d\theta$ are increasing equally, and when $\theta=\frac{1}{2}\pi$, dx is increasing $\frac{1}{2}\sqrt{2}$ times as fast as $d\theta$. Hence it is evident that a greater number of triangles exist for a certain length of arc in the vicinity of the vertex of the semicircumference whose diameter is the hypotenuse, than for the same length of arc near the origin C when the number of triangles is made a function of one of the legs of the triangle, and therefore Dr. Martin's conclusion is sound if we grant his assumption, namely, that the number of triangles is a function of one of the legs of the triangle.

But this assumption is what we refuse to grant. We believe that there are other triangles that are to be interpolated in the series in order that the totality of the triangles may be obtained and that these interpolated triangles are found by making the totality a function of the semicircumference.

From this consideration, it is evident that Dr. Martin's result, $\frac{1}{8}a^2$, is greater than the result, $\frac{a^2}{2\pi}$, which we are defending. The reason is, that according to his interpretation, the triangles are most numerous when $x=\frac{1}{2}\sqrt{2}a$, that is to say, when the vertex of the triangle coincides with the vertex of the semicircumference. Hence the sum of the areas of the triangles ought to be greater than when only as many triangles are taken in one portion of the arc of the semicircumference as in any other.

If the radius DB is made to revolve with uniform velocity about the point

D and its extremity B be joined with the points A and C then the totality of triangles will be formed and they will be uniformly distributed on the semicircumference whose diameter is the hypotenuse.

The question is not whether the triangles are uniformly distributed or not but what method gives the *totality* of the series.

Drury College, January 27, 1896.

NOTES.

ERRATA. Professor Beman calls my attention to a manifest error in Professor Klein's paper which I translated for the December number. Vol. II, page 350, should give the series $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.$ The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots.$ $=\log_e 2$ instead of $\frac{\pi}{4}$.

D. E. SMITH.

Dr. E. A. Bowser writes: Should not problem 43 [Calculus] read $\int_0^1 \frac{x^{a-1}-x^{-a}}{1+x}\,\frac{dx}{\log x}$, as in Price's Calculus, Vol. II, page 120?

NOTE ON THE SOLUTIONS OF PROBLEM 45, PAGES 274-75.

BY ARTEMAS MARTIN, LL. D.

There is but *one* case in Problem 45, Geometry, *as proposed*. Only the circumscribing circle is required.

The final result may be expressed in the more simple form

$$R=\frac{abc}{2\sqrt{[abc(a+b+c)]-(ab+ac+bc)}}.$$

In the second solution, page 275, the equation

"cos $BCA=\cos(BCA+BCO)$"

should be $\cos BCA=\cos(ACO+BCO)$.

EDITORIALS.

We shall be pleased to receive a catalogue from each of the schools and colleges where the MONTHLY is taken.

Charles De Medici, of New York City, writes, "Your magazine has certainly more merit than any other of the kind and ought to be well supported."

George W. Howe, Professor of Mathematics, State Normal School, Warrensburg, Mo., says, "The MONTHLY is a welcome visitor and I trust that you will continue it."

This number was delayed more than two weeks because of the failure of some proof reaching its destination. We feel confident that the March number will be mailed by the last of the month, and that thereafter the MONTHLY will appear regularly.

Cooper D. Schmitt, Professor of Mathematics, University of Tennessee, writes, "I enclose my subscription for the current year. I wish I had time to tell you how much I enjoy the MONTHLY and the good I get from it. It has caused me to study along certain lines that I had never before entered upon, and I feel that it does me an immense amount of good."

We are very thankful for the kind words that come from many of our contributors respecting the MONTHLY. That the work of getting out such a periodical every month is very arduous is not realized by all. That some errors creep into its pages is not surprising when every one thoroughly realizes the great work connected with the enterprise.

Professor E. P. Thompson, of Miama University, Oxford, Ohio, writes, "I send you with pleasure $2.00 for THE AMERICAN MATHEMATICAL MONTHLY for 1896. I get many a useful item, or point in discussion from it, and I hope you may prosper in the good work of putting into print the thoughts of the present workers in our beloved science."

Professor Thompson promises to contribute a paper on the "Mechanics of the Bicycle."

In *The Advance in Education* is an article, "A Class in Geometry under the Laboratory Plan," by Adelia R. Hornbrook, High School, Evansville, Ind. From this article we see that good practical use is made of the MONTHLY in the classroom. She says, "A group of boys, most of whom hope to go to the Polytechnic school, are working on a problem given them yesterday. They are much impressed with it, because it came out of the [*American*] *Mathematical Monthly*. They had never seen any mathematical publications except the text books, and the *Monthly*, with its intricate diagrams, mysterious figures, and unfamiliar terms was a revelation to them." There are thousands of teachers in our High

Schools that could most profitably follow the writer of the above article's plan. Not necessarily that the MONTHLY be used but that the teachers of mathematics carry the spirit of the great living subject into their classrooms. There is no better way to do this than for every teacher to take some good magazine especially devoted to his favorite study.

BOOKS AND PERIODICALS.

Elements of the Differential and Integral Calculus with Examples and Practical Applications. By J. W. Nicholson, A. M., LL. D., President and Professor of Mathematics, Louisiana State University and Agricultural and Mechanical College. 8vo. Cloth, 256 pp. New York and New Orleans: University Publishing Co.

We have long been expecting this unique work on the Calculus as Col. Nicholson apprised us more than a year ago that he was preparing a work on the subject which he expected would create a stir among mathematicians. In this, I think he will realize his expectation, as his work is a great departure from the long beaten path of the traditional Calculus. None of the metaphysical speculations of Newton, Leibinz, Carnot, D'Alembert, Berkeley, Duhamel, Cavalieri, Marquis de L'Hopital, etc., are met with, in reading this book. The idea is simply an extention of mathematical principles without assuming vague metaphysical propositions.

The chief distinctions of this treatise is that, (1) it is based on the conception of *Proportional Variations*, (2) the treatment of dx as a variable, (3) a rigorous deduction of simple tests of absolute convergency, without recourse to the remainder in Taylor's formula, (4) an extension of the ordinary rules for finding maxima and minima, (5) a chapter on Independent Integration, (6) integration by independent coefficients, (7) the introduction of *turns* in curve tracing, and (8) a new proof of Taylor's formula.

The treatment of dx as a variable is the only rational way of viewing dx as a quantity at all. We do not think that Col. Nicholson has wandered too far from the usual method of treating the subject and we are sure the beginner in Calculus will hail the work with joy.

It is time for the Calculus to be treated on sound mathematical principles and not those of metaphysics. We very gladly recommend this new work to the favorable consideration of teachers and students desiring a good text book on the Calculus. B. F. F.

The Science Absolute of Space. Independent of the truth or falsity of Euclid's Axiom XI (which can never be proved *a priori*). By John Bolyai. Translated from the Latin by Dr. George Bruce Halsted, President of the Texas Academy of Science. Fourth edition. Vol. three of the Neomonic Series. Published at the Neomon, 2407 Guadalupe Street, Austin, Texas. Cloth, 71 pp. Price, $1.00.

Dr. Halsted has just got out the fourth edition of his translation of Bolyai's "Science Absolute of Space." The work is enriched by many interesting particulars concerning the lives of the celebrated author of the Non-Euclidean Geometry, Bolyai Janos, and his father, Bolyai Farkas. This little work is worth a careful reading at least once a year.

B. F. F.

Concrete Geometry for Beginners. By A. R. Hornbrook, A. M., Teacher of Mathematics in High School, Evansville, Indiana. 12mo. Cloth, 201 pp. Price, 75 cents. New York and Chicago: American Book Co.

This book is designed as an Introductory Course to the Study of Demonstrative Geometry. The author is a very thorough and efficient teacher of mathematics, and is intensely interested in the subject. The book is carefully and skillfully written, and can not be too highly recommended for the place it is designed to occupy. Were all students carefully instructed by the Laboratory Method, in Concrete Geometry, better results would be obtained while studying Demonstrative Geometry. B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York City.

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